FINITELY ADDITIVE ZERO-ONE LAWS

by

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Technical Report #386

March 1981

*Research supported by National Science Foundation Grant MCS77-28424.

Abstract

Finitely additive versions of the Lévy and Kolmogorov zero-one laws are stated and proved. An example is given to show that an analogous version of the Hewitt-Savage zero-one law is false.

Key words and phrases. Finitely additive probability, extension of measures, topological measures, finitely additive stochastic processes, theory of gambling.

American Mathematical Society subject classifications. 60G05, 60A05, 60F15, 60G45, 60B05.

1. Introduction

A finitely additive theory of integration on countable product spaces was developed by Dubins and Savage (1974, 1976) and extended by Purves and Sudderth (1976). Here the main results are finitely additive versions of the Kolmogorov zero-one law and the Lévy zero-one law. A counterexample is given to an analogous version of the Hewitt-Savage zero-one law.

2. Preliminaries

Let X be an arbitrary non-empty set and X^* the set of all finite sequences of members of X, including the empty sequence. The basic ingredients in what follows are the set X and an indexed family $\sigma = (\sigma(q);\ q \in X^*)$ of finitely additive probability measures defined on all subsets of X. Informally, such a family σ can be used to generate an infinite sequence of random members of X: first the probability measure σ(empty sequence) is used to choose an x_1; then $\sigma(x_1)$ is used to choose an x_2; then $\sigma(x_1,x_2)$ is used to choose an x_3; and so on. In the theory of gambling developed by Dubins and Savage the family σ is called a <u>strategy</u>. As described in Dubins and Savage (1976, pages 7-21), Dubins (1974) and Purves and Sudderth (1976), each strategy σ determines a finitely additive probability measure on the sigma-field $\mathcal{B}$ of subsets of $H = X \times X \times \ldots$ which is generated by the open subsets of H when X is assigned the discrete topology and H the product topology. This probability measure is also denoted by σ and is regarded as the distribution of the random sequence $x_1, x_2, \ldots$.

The four clauses below summarize the main features of this finitely additive theory needed to obtain the two zero-one laws. The proofs may be found in the references. The first two clauses are simple regularity properties:

(i) If $0 \subseteq H$ is open,

$$\sigma(0) = \sup\{\sigma(K);\ K \subseteq 0,\ K \text{ clopen}\}.$$

(ii) If $B \in \mathcal{B}$ and $\varepsilon > 0$ is given, there is a closed set C and an open set 0 such that $C \subseteq B \subseteq 0$ and $\sigma(0 - C) < \varepsilon$.

The next clause gives a tractable sufficient condition for a countable union of sets to have measure zero. The condition might be stated loosely as follows. Let $x_1, x_2, \ldots$ be generated by σ as described above. Temporarily, call a set N in $\mathcal{B}$ <u>always null</u>, if for each $p \in X^*$, the conditional probability of $x_1, x_2, \ldots$ falling in N, given the finite past p (i.e., given $x_j = p_j$ for $j \leq$ number of terms in p) is zero. Then a sufficient condition for a countable union $N^1 \cup N^2 \cup \ldots$ to have σ-measure zero is that each N^i be always null. To state this condition precisely requires a little notation, together with the idea of conditional strategy. To begin with the notation, let $p, q \in X^*$ and $h \in H$. Then pq is the member of X^* whose terms consist of the terms of p followed by the terms of q; and ph is the member of H whose terms consist of the terms of p followed by the terms of h. The <u>p-section</u> of a set $E \subseteq H$ is the set $\{h \in H\colon ph \in E\}$ and is denoted by Ep. If $\sigma = (\sigma(q)\colon q \in X^*)$ is a strategy, the <u>conditional strategy given</u> p, written $\sigma[p]$, is defined by $\sigma[p](q) = \sigma(pq);\ q \in X^*$. If $E \in \mathcal{B}$, the quantity $\sigma[p]Ep$, which it is natural to regard as the conditional probability that $x_1, x_2, \ldots$ falls in E given the finite past p, will be written $\sigma(E|p)$.

(iii) Let $N^1, N^2, \ldots \in \mathcal{B}$. If $\sigma(N^i|p) = 0$ for all $p \in X^*$ and all $i = 1,2,\ldots$ then $\sigma(N^1 \cup N^2 \cup \ldots) = 0$.

Finally, if s is a stop rule and $h \in H$, set $p_s(h) = (h_1 \ldots, h_m) \in X^*$, where $m = s(h)$; and set $\sigma(E|p_s)(h) = \sigma(E|p_s(h))$. Then

(iv) If $E \in \mathcal{B}$,

$$\sigma(E) = \int \sigma(E|p_s)d\sigma.$$

3. <u>Kolmogorov zero-one law</u>

A family $\sigma = (\sigma(q); q \in X^*)$ of finitely additive probability measures defined on all subsets of X is said to be <u>independent</u> if $\sigma(p) = \sigma(q)$ whenever p,q have the same number of terms. A set $B \subseteq H$ is said to be a <u>tail set</u> if $Bp = Bq$ whenever p,q have the same number of terms.

<u>Theorem 1</u>. If σ is independent and $B \in \mathcal{B}$ is a tail set then $\sigma(B)$ is zero or one.

<u>Proof</u>. First note that for all $p \in X^*$ $\sigma(B|p) = \sigma(B)$. To see this, fix a $p \in X^*$ and suppose p contains n terms. Then the conditional strategies $\sigma[p_n(h)]$, $h \in H$ are all equal to $\sigma[p]$, by the independence of σ. Further $Bp_n(h) = Bp$, all $h \in H$. Applying (iv) of section 2, $\sigma(B) = \int \sigma(B|p_n(h))d\sigma = \sigma(B|p)$.

Now let $\varepsilon > 0$. Invoking (i), (ii) of section 2, choose 0 open such that $0 \geq B$, $\sigma(0) < \sigma(B) + \varepsilon$ and K clopen such that $K \subseteq 0$,

$\sigma(K) > \sigma(0) - \varepsilon$. Then

$$\begin{aligned}\sigma(B) &= \sigma(B \cap K) + \sigma(B \cap K^c)\\ &\leq \sigma(B \cap K) + \sigma(0 \cap K^c)\\ &\leq \sigma(B \cap K) + \varepsilon.\end{aligned}$$

Since K is clopen, there is a stop rule s such that K is determined by time s (Dubins and Savage, Theorem 2.7.1 and Corollary 2.7.1). For that s, calculate as follows:

$$\begin{aligned}\sigma(B \cap K) &= \int \sigma(B \cap K|p_s)d\sigma\\ &= \int_K \sigma(B|p_s)d\sigma\\ &= \int_K \sigma(B)d\sigma\\ &= \sigma(B)\sigma(K).\end{aligned}$$

Hence,

$$\begin{aligned}\sigma(B) &\leq \sigma(B)\sigma(K) + \varepsilon\\ &\leq \sigma(B)\sigma(0) + \varepsilon\\ &\leq (\sigma(B))^2 + 2\varepsilon.\end{aligned}$$

It follows that $\sigma(B) = 0$ or 1.

The next result is rather curious.

<u>Theorem 2</u>. If σ is independent, the measure σ is countably additive when restricted to the collection of tail sets in $\mathcal{B}$.

<u>Proof</u>. Let $B^1, B^2, \ldots$ be tail sets and in $\mathcal{B}$. Assume $\sigma(B^i) = 0$ for all i. By the preceding theorem, it suffices to show $\sigma(B^1 \cup B^2 \cup \ldots) = 0$. Fix $p \in X^*$. Since $\sigma(B^i|p) = \sigma(B^i)$ and $\sigma(B^i) = 0$, the conclusion

follows from (iii) of section 2.

4. The Lévy zero-one law.

This is stated as follows:

Theorem 1. Let $B \in \mathcal{B}$. Then

$$\sigma\{h \in H: \sigma(B|p_n(h)) \to 1_B(h)\} = 1.$$

The proof rests on the following lemma, which is adapted from Lévy's (1937) original argument.

Lemma 1. Let $K \subseteq H$ be clopen, $B \in \mathcal{B}$. Set

$$\rho_n(h) = |\sigma(B|p_n(h)) - 1_K(h)|, \; h \in H.$$

Then, if $\alpha > 0$,

$$\sigma\{h: \limsup \rho_n(h) > \alpha\} \leq \sigma(B \Delta K)/\alpha.$$

Proof. Let $\varepsilon > 0$. Let K be determined by time s. Define t on H by

$$\begin{aligned} t(h) &= \text{first } n \text{ (if any) such that } n \geq s(h) \text{ and } \rho_n(h) > \alpha \\ &= \infty \text{ if there is no such } n. \end{aligned}$$

The set $[t < \infty]$ is open so there is a clopen $J \subseteq [t < \infty]$ for which $\sigma(J) > \sigma[t < \infty] - \varepsilon$. Then there is a stop rule r which agrees with t on J (Dubins and Savage, Theorem 2.11.1). Set $L = [t = r]$ and observe that $J \subseteq L \subseteq [t < \infty]$.

Calculate as follows:

$$\sigma(B \Delta K) \geq \sigma((B \Delta K) \cap L).$$

Set $A = (B \Delta K) \cap L$. Then

$$\sigma(A) = \int \sigma(A | p_r(h)) d\sigma(h).$$

Since L is determined by time r the integrand can be re-written to obtain

$$\begin{aligned}\sigma(A) &= \int 1_L(h) \sigma(B \Delta K | p_r(h)) d\sigma(h) \\ &\geq \int 1_L(h) | \sigma(B | p_r(h)) - \sigma(K | p_r(h)) | d\sigma(h) \\ &\geq \alpha\sigma(L).\end{aligned}$$

The last inequality follows from the fact that for $h \in L$, $r(h) = t(h)$.

The calculation implies $\sigma[t < \infty] \leq \sigma(B \Delta K)/\alpha$, which, in turn, implies the lemma.

Lemma 2. For $h \in H$, set

$$d(h) = \lim\sup | \sigma(B | p_n(h)) - 1_B(h) |.$$

Then, if $\alpha > 0$ and K is clopen

$$\sigma\{h \in H | d(h) > \alpha\} \leq 4\sigma(B \Delta K)/\alpha.$$

Proof. Let K be clopen. Adding and subtracting $1_K(h)$ within the absolute value shows $d(h)$ is at most the sum of the terms $\lim\sup \rho_n(h)$, $|1_K(h) - 1_B(h)|$. So if d exceeds α, at least one of the two terms must exceed $\alpha/2$. The rest is straightforward in view of Lemma 1.

Proof of Theorem 1.

By (i), (ii) of section 2 there is a clopen set K such that

$\sigma(B\Delta K)$ can be made arbitrarily small. Then Lemma 2 implies that $\sigma[d > \infty] = 0$. If σ were countably additive, this would be enough to prove the theorem. In the finitely additive framework assumed here, it suffices to show $\sigma(d > \alpha|p) = 0$ for all $p \in X^*$. This is (iii) of section 2--set $N^i = [d > 1/i]$, $i = 1,2,\ldots$ to get $\sigma[d > 0] = 0$.

To establish that $\sigma(d > \alpha|p) = 0$ return to the setting of Lemma 2. Note that the function d defined there depends on B and σ, so that it could be written $d(\cdot;B,\sigma)$ to show the dependence explicitly. Now

$$\{h \in H|d(h;B,\sigma) > \alpha\}p = \{h' \in H|d(h';Bp,\sigma[p]) > \alpha\}.$$

Thus Lemma 2, with $\sigma[p]$, Bp in place of σ, B respectively shows that $\sigma(d > \alpha|p) = 0$. This completes the proof.

Finally, the Hewitt-Savage zero-one law does not hold in the finitely additive framework used here. For an example, let X be the set of positive integers and γ a finitely additive measure defined on all subsets of X. Let $\sigma(p) = \gamma$, for all $p \in X^*$. Informally, σ corresponds to using γ repeatedly to generate an infinite sequence of positive integers. Suppose γ has the additional property that $\gamma(A) = 0$ for all finite sets $A \subseteq X$. Then the σ-measure of the set $\{h \in H|h_1 < h_2 < \ldots\}$ is equal to one. (A proof can be based on Lemma 7.1 of Purves and Sudderth (1976). In the notation of that lemma, take $K_n = [h_1 < h_2 < \ldots < h_n]$, $r_n = n$, and $\alpha_n = 1$.) It follows that $\sigma\{h \in H|\min_i h_i \text{ is even}\} = \gamma\{\text{even numbers}\}$. This contradicts the Hewitt-Savage law because the set on the left is permutation-invariant.

S. Ramakrishnan (1980) has, however, shown that countable intersections of permutation invariant open sets must have σ measure zero or one.

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